Department of Applied Mathematics

Card trick

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Math Exercises for You, June 22, 2017, VŠB

Description
Maximum
Solution might exist
Solution exists
Finding a solution
Solution

How the trick goes

We have a deck of cards:

- assistant lets a volunteer from the audience to choose 5 cards at random

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. . . let's try!

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Total $5 \cdot 24 + 4 = 124$ cards.

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This maximum can be achieved!

Our deck has cards 1, 2, ..., 124.

Existence of the solution not excluded

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Number of 5-tuples among $n = 124$ cards is

$$f = \binom{124}{5} = \frac{124 \cdot 123 \cdot 122 \cdot 121 \cdot 120}{120}$$

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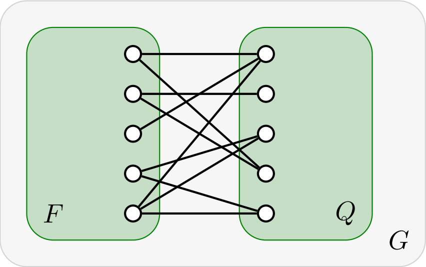
To every (ordered) quadruple (a_1, a_2, a_3, a_4) there exists a unique (unordered) 5-tuple $\{a_1, a_2, a_3, a_4, x\}$
(and vice versa).

Solution exists

Marriage Theorem

Let G be a bipartite graph with partite sets F and Q. In graph G there is a perfect matching M, which saturates all vertices in the set F if and only if $|S| \leq |N(S)|$ for every subset $S \subseteq F$.



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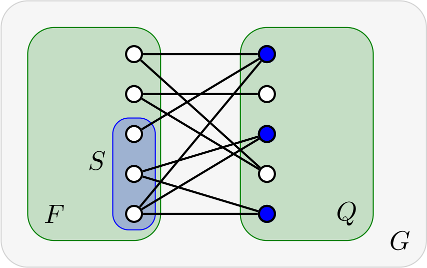

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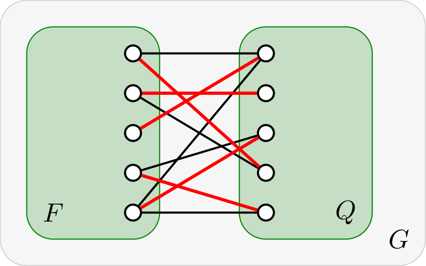

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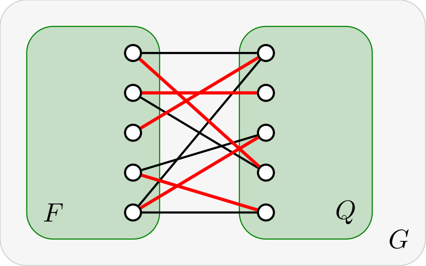

Corollary

Every regular bipartite graph with at least one edge has a perfect matching.

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Each (ordered) quadruple is joined by an edge to 120 5-tuples (in which this quadruple is included).

Similarly, each (unordered) 5-tuple is joined to 120 quadruples (that can be obtained from it).
See: $\binom{5}{4} = 5$ quadruples, each in $P(4) = 24$ different arrangements.

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G is a 120-regular bipartite graph and by (corollary of) Marriage Theorem G has a perfect matching M.
It is enough to remember $|M| = 225,150,024$ edges of the matching.

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Bipartite graph G with partite sets F a Q.

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If we draw the graphs on a large canvas so that for each vertex we use at least a square inch (120 adjacent edges), how large the canvas would be?

How many square feet? Square yards? Square miles?

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No problem finding the matching by a computer.
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... and here we are, we can do the trick!

Thank you for your attention.

Solution

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Nobody can learn these $225,150,024$ pairs by heart!

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There is a nicer and simpler solution (of course).

"It keeps me from looking at my phone every two seconds." Liam Walsh/The New Yorker

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Note about congruences

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Congruent numbers

Two integers a, b are *congruent modulo n* if they have the same remainder after (integer) division by positive integer n. The integer n is the *modulus*.

We write $a \equiv b \pmod{n}$.

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- difference $a - b$ is a multiple of n
- number b can be obtained from a by adding a of n.

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Often it is convenient to regard congruent integers as "same" or "equal" (with respect to integer division by n).

Actual solution

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We denote the five by $x_0 < x_1 < x_2 < x_3 < x_4$.
We denote $s = x_0 + x_1 + x_2 + x_3 + x_4$.

The assistant evaluates $i \equiv s \pmod 5$, $i \in [0, 4]$.

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We denote by $y = x - i$, where $y \in [1, 120]$ the rank of the missing card.

Actual solution (continued)

The following congruences hold

$$\begin{aligned} x - y &\equiv i \pmod 5 \\ x - y &\equiv s \pmod 5). \end{aligned} \quad (1)$$

We denote $r = x_0 + x_1 + x_2 + x_3 + x_4 - x_i = s - x$.

$$x = s - r. \quad (2)$$

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$$x = s - r. \quad (2)$$

By substitution (2) to (1) we get the following

$$\begin{aligned} s - r - y &\equiv s \pmod 5 \\ -r - y &\equiv 0 \pmod 5 \\ -r &\equiv y \pmod 5. \end{aligned} \quad (3)$$

The key observation is, that rank y of the missing card is congruent to $-r$ modulo 5.

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Among the remaining 120 cards there are $120/5 = 24$ congruent to $-r \pmod 5$.
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Here the assumption of choosing card x_i helps.
By this assumption we determine i (number of cards smaller than y).
And so the magician can determine $x = y + i$.

Example

The volunteer chooses cards $12, 37, 38, 90, 105$.

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The assistant evaluates $s = 12 + 37 + 38 + 90 + 105 \equiv$
$\equiv 2 + 2 + 3 + 0 + 0 \equiv 2 \pmod 5$.

The assistant picks card $x_2 = 38$ to give to the volunteer and evaluates $y = x_i - i = 38 - 2 = 36$.

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Now $y = 36 = 7 \cdot 5 + 1 = (7 + 1) \cdot 5 + (1 - 5) = 8 \cdot 5 - 4$, thus the "quotient" is $k = 8$.
The four cards have to be ordered as $37, 12, 105, 90$, which is the 8th permutation.

24 permutations

k	permutation	k	permutation
1	1 2 3 4	13	3 1 2 4
2	1 2 4 3	14	3 1 4 2
3	1 3 2 4	15	3 2 1 4
4	1 3 4 2	16	3 2 4 1
5	1 4 2 3	17	3 4 1 2
6	1 4 3 2	18	3 4 2 1
7	2 1 3 4	19	4 1 2 3
8	2 1 4 3	20	4 1 3 2
9	2 3 1 4	21	4 2 1 3
10	2 3 4 1	22	4 2 3 1
11	2 4 1 3	23	4 3 1 2
12	2 4 3 1	24	4 3 2 1

Table: Table of permutations of four elements.

Example (continued)

Now the magician observes the four cards.
Since $r = 12 + 37 + 90 + 105 \equiv 2 + 2 + 0 + 0 = 4$, he knows that $y \equiv -4 \pmod 5$.

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$$y = 5k - r = 5 \cdot 8 - 4 = 36.$$

Because $36 \geq 12$ and $36 + 1 \geq 37$, the thirty sixth missing card is $x = y + 2 = 38$.

This is how the magician "knows" it is the card 38 the volunteer has in his/her hand.

Generalizations

What if the volunteer picks n cards (not necessarily 5)?

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n	maximum number of cards $n! + (n-1)$
2	3
3	8
4	27
5	124
6	725
7	5 046
8	40 327
9	362 888
10	3 628 809

Beware!
Computations have to run modulo n (so far modulo 5 easy).

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  - lets one volunteer to pick the five cards,
  - then lets another volunteer to choose one of these cards (provided she can see the value)?
- the assistant would choose two cards to give back?

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