SIMPLE MATHEMATICAL MODELS WITH A VERY RICH DYNAMICS

Marek Lampart

Department of Applied Mathematics
&
IT4Innovations
VŠB – Technical University of Ostrava

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Meaning of the word „Chaos"

chaos | $\kappa\epsilon\iota os$ |
noun [mass noun]
complete disorder and confusion: *snow caused chaos in the region*.

- *Physics* the property of a complex system whose behaviour is so unpredictable as to appear random, owing to great sensitivity to small changes in conditions.
- the formless matter supposed to have existed before the creation of the universe.
- (**Chaos**)*Greek Mythology* the first created being, from which came the primeval deities Gaia, Tartarus, Erebus, and Nyx.

ORIGIN late 15th cent. (denoting a gaping void or chasm, later formless primordial matter): via French and Latin from Greek **khaos 'vast chasm, void'**.

Application of dynamical systems

- In the fields of
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  - research
  - engineering
- Scientific fields
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  - art
  - genetics
  - teology
  - physics
  - chemistry
  - biology
  - economy
  - politology
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  - mechanics
  - electrotechnics
  - geoinformatics
  - lingvistics
  - medicine

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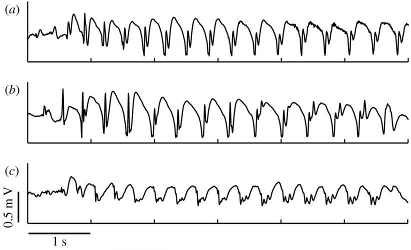
$\vdots$

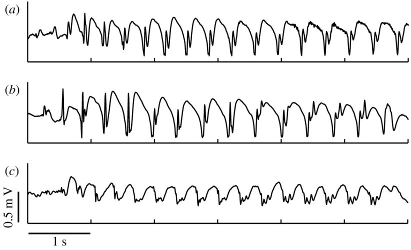
Is chaos necessarily bad?



Marten, F. et all, *Onset of polyspike complexes in a mean-field model of human electroencephalography and its application to absence epilepsy*, Phil. Trans. Royal Soc. A: Math., Phys. and Eng. Sci., vol. 367, no. 1891, pgs. 1145-1161, 2009.

The irregular and ragged-looking one is a normal human, eyes-open EEG, and the beautiful periodic one is an epileptic seizure!

Is chaos necessarily bad?



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Elementary example - deposit interest

P_0 — the initial value of the deposit, r — rate interest

$$P_1 = P_0 + rP_0 = (1+r)P_0$$
$$P_2 = P_1 + rP_1 = (1+r)P_1 = (1+r)^2 P_0$$
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Question no.1: How long it will take to double the initial value? (r=0,02)
Question no.2: What will be the answer to the first question if we take into the consideration in each step the interest only of the initial value?

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Difference equation and iterations

The process in which the predecessing state affects the next one is described by the *difference equations*

$$x_{n+1} = f(x_n).$$

This equation describes how to repeat some operation many times repeatedly. This is called as *iteration*.

Definition

Let $f : X \to X$ is a continuous map of a compact metric space X. Then

$$f^n(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n-\text{times}}(x) = \underbrace{f(f(f \cdots f}_{n-\text{times}}(x)) \cdots)$$

is called as n**-th fold iteration of the point** x **under the map** f, where $n \in \mathbb{N}$.

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is called as **n-th fold iteration of the point x under the map f**, where $n \in \mathbb{N}$.

Dynamical system

Definition

Let $f : X \to X$ be a continuous map on a compact metric space X. Then the ordered pair

$$(X, f)$$

is called **(discrete) dynamical system.**

Function examples generating difference equations

TABLE 1. SOME FIRST-ORDER DIFFERENCE EQUATIONS, $x_{t+1} = F(x_t)$, TAKEN FROM THE BIOLOGICAL LITERATURE, WHICH CAN EXHIBIT CHAOTIC DYNAMICS

$F(x)$	source
$x \exp[r(1-x)]$	Moran (1950), Ricker (1954), Macfadyen (1963), Cook (1965), Pacala & Silander (1985)
$x[1+r(1-x)]$	Maynard Smith (1986), May (1972), Li & Yorke (1975)
λx, if $x<1$ λx^{1-b}, if $x>1$	Haldane (1953), Varley *et al.* (1973) and references therein
$\lambda x/(1+ax^b)$	Maynard Smith (1974), Bellows (1981)
$\lambda x(1+ax)^{-b}$	Hassell (1974)
$x[1/(a+bx)-\sigma]$	Utida (1957)
$\lambda_+ x$ if $x<1$ $\lambda_- x$, if $x>1$	Williamson (1974), with $\lambda_+ > 1$, $\lambda_- < 1$
$\lambda x[1-I(x)]$	May (1985), with $I(x)$ given by $1-I = \exp(-Ix)$
$\lambda x \mathrm{e}^{-x} \sum_{i=0}^{\infty} \frac{x^i}{i!(1+\alpha i)}$	Pacala & Silander (1985), Crawley & May (1987)
$\frac{\lambda x}{(1+ax)^b + cx}$	Watkinson (1980)

R.M. May. *Chaos and the Dynamics of Biological Populations*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, **413**(1987) s. 27–43.

Generic example - The system of logistic maps

$$F_\mu(x) : [0,1] \to [0,1] \qquad F_\mu(x) = \mu x(1-x)$$

R.M. May. *Simple mathematical models with very complicated dynamics*, Nature, **261**(1976) s. 459–467.

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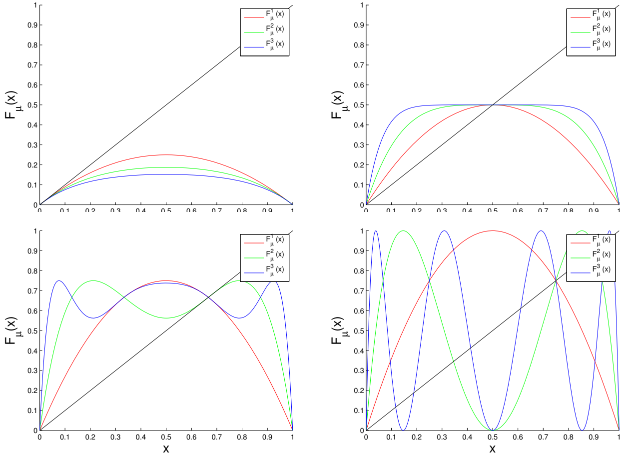


Periodicity

Definition

Let (X, f) be a dynamical system. The point $x \in X$ is called **fixed** iff

$$f(x) = x.$$

The point $x \in X$ is called **periodic with a prime period** n iff

$$f^n(x) = x$$

and $f^m(x) \neq x$ for any $0 < m < n$.

Generic example - The system of logistic maps

Theorem

For $F_\mu(x) = \mu x(1-x)$ defined on $[0,1]$ it holds:

1. $F_\mu(0) = 0$ and $F_\mu(p_\mu) = p_\mu$, where $p_\mu = (\mu - 1)/\mu$,
2. if $1 < \mu \leq 4$ then $0 < p_\mu < 1$,
3. if $\mu = 1$ then $\mathrm{Fix}(F_1) = \{0\}$,
4. if $0 < \mu < 1$ then $p_\mu \notin [0,1]$.

Generic example - The system of logistic maps

Theorem

For $0 < \mu < 1$ and every $x \in [0,1]$ it holds: $\lim_{n\to\infty} F_\mu^n(x) = 0$.

Theorem

Let $\mu > 1$, then

1. if $x < 0$ then $\lim_{n\to\infty} F_\mu^n(x) = -\infty$,
2. if $x > 1$ then $\lim_{n\to\infty} F_\mu^n(x) = -\infty$.

Theorem

Let $1 < \mu < 3$ and $0 < x < 1$ then $\lim_{n\to\infty} F_\mu^n(x) = p_\mu$.

Periodicity

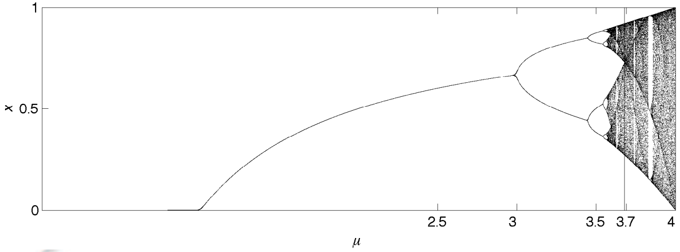
Theorem

Let $([0,1], f)$ be a dynamical system with 3-periodic point. Then f has periodic points of all periods.

A.N. Šarkovskij. *O ciklach i strukture nepreryvnogo preobrazovanija*. Ukrain. Mat. Žurnal 17.3 (1965), s. 104–111.

Generic example - The system of logistic maps

Bifurcation diagram



Chaos in the sense of Devaney

Definition

The dynamical system (X, f) is **sensitive on initial conditions**, if there is $\delta > 0$ such that for every $x \in X$ and a neighbourhood $B(x, \epsilon)$ there is $y \in B(x, \epsilon)$ and $n \in \mathbb{N}$ such that

$$d(f^n(x), f^n(y)) > \delta.$$

Chaos in the sense Devaney

Definition

Let (X, f) be a dynamical system. The map f is called **chaotic in the sense of Devaney** iff:

1. f is topological transitive, i.e. f has dense orbit in X;
2. (X, f) is sensitive on initial conditions;
3. the set of all periodic points of f is dense in X.

R.L. Devaney. *An Introduction to Chaotic Dynamical Systems*. 2nd.Menlo Park, CA: Addison-Wesley, 1989.

Chaos in the sense of Li and York

Definition

Let (X, f) be a dynamical system. The map f is called **chaotic in the sense Li and York** iff there is uncountable set $S \subset X$ such that for any different $x, y \in S$ (that is $x \neq y$) the following conditions are held:

1. $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$
2. $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0.$

The set S is called *scrambled*.

T.-Y. Li a J. A. Yorke. *Period three implies chaos*. Amer. Math. Monthly, **82** (1975), s. 985–992.

Chaos in the sense Li and York

Theorem

Let $([0,1], f)$ be a dynamical system with 3-periodic point. Then f chaotic in the sense of Li and York.

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Thank you for attention